It can be shown from the definition of ϕ and Eq. (8) that

$$D - N/(2 + N) = 2T_0/[(2 + N)Te] \qquad (18)$$

Through the use of the above, and the fact that the total temperature, T_0, is constant throughout the interaction, Eq. (17) becomes

$$(1 + N_2)/(N_2^2 + 2N_2) = \alpha \qquad (19)$$

where

$$\alpha = \frac{(1 + N_1)}{(N_1^2 + 2N_1)} \frac{Me_1}{Me_2} \left(\frac{Te_2}{Te_1}\right)^{1/2} \cos\theta \qquad (20)$$

Explicit solution of Eq. (19) for the quantity N_2 yields

$$N_2 = (1 + 0.25\alpha^{-2})^{1/2} + 0.50\alpha^{-1} - 1 \qquad (21)$$

Equations (21) and (20) determine N_2 from known quantities in the region of interaction. The corresponding compressible power-law reciprocal exponent, n_2, may be determined from N_2 and M_{e2} through the use of Fig. 1. The boundary-layer thickness ratio δ_2/δ_1 is then obtained from Eq. (14) or (16) through the use of Eq. (7).

Analytical Results

Sample analytical results are presented in Fig. 3. The ratios of δ_2/δ_1 and N_2/N_1 are presented as a function of N_1 for a range of freestream Mach numbers (M_{e1}) and a compression corner angle of 10°. It can be seen from Fig. 3 that the predicted effect of the interaction is to increase the boundary layer velocity-profile exponent ($1/n$), which essentially "weakens" the boundary layer and increases its susceptibility to separation. The results indicate a net reduction in boundary layer thickness due to the increase in density of the fluid in the boundary layer as it is influenced by the compression field downstream of the shock wave. Although no experimental data are available to verify these analytical results for this type of shock/boundary-layer interaction, data are presented in Refs. 4 and 5 which clearly indicate similar boundary layer behavior in incident-reflecting-shock/boundary-layer interactions in cases where separation is not evident.

It should be emphasized that these theoretical results do not account for real viscous effects, and that boundary-layer separation is a significant or dominating factor in flow situations where it is present. The theoretical results of this paper should not be applied in cases where the limits of incipient boundary-layer separation have been reached. Experimental data on incipient separation for this kind of interaction were obtained by Kuehn[6] for boundary layer Reynolds numbers ($R_{e\delta}$) of from 1.5 to 7.5 $\times 10^4$. Data for Reynolds numbers ($R_{e\delta}$) of from 1.5 to approximately 7 $\times 10^6$ are presented in Ref. 2. An approximation for the range of Reynolds numbers considered for Mach numbers up to 4.0 is the following:

$$\theta \text{ incipient separation} \geq (5.5M_{e1})^\circ \qquad (22)$$

The aforementioned references should be consulted if θ exceeds 5.5 Me_1.

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Curvature Effects in the Laminar and Turbulent Freejet Boundary

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Nomenclature

C	= normalized curvature parameter
E	= entrainment parameter
f	= normalized stream function
k	= curvature of the "major streamline"
k_0	= curvature parameter
p	= pressure
R	= radius of curvature of the major streamline
u	= x component of velocity
U_1	= x component of velocity along major streamline
U_{max}	= maximum x component of velocity
v	= y component of velocity
x	= coordinate along major streamline
y	= coordinate normal to major streamline
ϵ	= eddy viscosity
η	= normalized y coordinate
κc	= constant in the expression for eddy viscosity
μ	= viscosity
ν	= kinematic viscosity
ρ	= density
σ	= scaling factor in normalized y coordinate
τ	= shearing stress
ψ	= stream function

IF a nearly uniform stream passes from a nozzle, say, into a stagnant region there is a mixing between the jet issuing from the nozzle and the stagnant fluid. Just downstream of the point where the jet enters the stagnant fluid the flowfield is described by the so-called "freejet boundary." Normally for such flows there is no pressure gradient across the jet and the streamlines have no curvature other than that associated with normal spreading of the jet. There are physical cases in which there is a pressure gradient across the flow and as a result the streamlines are curved. Perhaps the most timely example of such a curved freejet boundary occurs in a fluid amplifier device which employs the "Coanda" effect. In such a device the jet entrains fluid between the main body of the jet and a wall next to the jet, resulting in a reduction of the pressure on the side of the jet next to the wall. The reduced pressure causes a deflection of the jet toward the wall until an equilibrium situation results.

A related physical situation occurs in the jet flap[1] where a jet at one velocity issues from the trailing edge of an airfoil and mixes with parallel streams of different velocities. A curved jet results in which a pressure difference is maintained by the curved jet which involves not one but two freejet boundaries.

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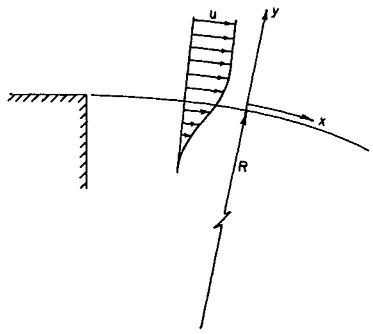

Fig. 1 Coordinate system.

Since curvature (and pressure gradient) may affect the entrainment of the jet it is important to investigate the effects of curvature. In the present paper an investigation is made of the effects of curvature on the freejet boundary between a curved irrotational flow and a stagnant fluid. It is shown that similar solutions are possible in both the laminar and turbulent cases for certain variations of jet curvature, provided the curvature is moderate. In fact, for both the laminar and turbulent cases the equations of motion can be reduced to a single ordinary differential equation, the solution of which is presented herein.

The results of this solution indicate that the velocity profile within the freejet boundary is, for all practical purposes, independent of the curvature. The effects of the pressure across the freejet boundary on the curvature and on the entrainment are also determined.

The problem considered here represents only a first approximation to the two physical situations noted above. In both the Coanda effect and the jet flap the physical situation is so complicated that a nonsimilar analysis of the flowfield is required. Apparently, however, there are no solutions, similar or otherwise, to curved jet type flowfields. It is hoped that the present similar solution to a simple flowfield will yield physical insight into the problem of curved jet type flows and will thus serve as a guide in attacking the more complicated problem.

Analysis

Consider the two-dimensional, steady incompressible fluid motion described in the curvilinear coordinate system shown in Fig. 1. In the present case of the freejet boundary the x axis is taken to be along one of the streamlines in the flowfield. This streamline is denoted here as the major streamline. It should be noted that in this coordinate system the major streamline is the only streamline which coincides with a y = constant curve. It is assumed that, as in the no curvature case, the motion of the curved freejet boundary is adequately described by the boundary layer equations. For moderate curvature and in the coordinate system under consideration these equations are[2]

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{1}$$

$$u\partial u/\partial x + v\partial u/\partial y = -(1/\rho)\partial p/\partial x + (1/\rho)\partial \tau/\partial y \tag{2}$$

$$ku^2 = (1/\rho)\partial p/\partial y \tag{3}$$

where for laminar flow $\tau = \mu\partial u/\partial y$ and for turbulent flow the laminar shear is neglected and the shearing stress is taken as $\tau = \epsilon\partial u/\partial y$. Here ϵ is the eddy viscosity and k is the curvature of the major streamline. Eliminating the pressure gradient from Eq. (2) by cross differentiation of Eqs. (2) and (3) yields

$$u\frac{\partial^2 u}{\partial x\,\partial y} + v\frac{\partial^2 u}{\partial y^2} = -2ku\frac{\partial u}{\partial x} - u^2\frac{dk}{dx} + \frac{1}{\rho}\frac{\partial^2 \tau}{\partial y^2} \tag{4}$$

In the present problem it is assumed that for large negative values of y the fluid is stagnant so that

$$\lim_{y\to-\infty} u(x,y) = 0 \tag{5}$$

On the other hand for large positive values of y the fluid motion is assumed to be irrotational. For moderate curvature the condition of irrotationality is given by[2]

$$(1/u)\partial u/\partial y = -k \qquad (\text{for } y \to +\infty) \tag{6}$$

Since $y = 0$ defines the "major" streamline we have along this streamline in the coordinate system shown

$$v(x,0) = 0 \tag{7}$$

Laminar flow

For laminar flow it is easily shown that similar solutions to the equations of motion which reduce to the standard solution when $k = 0$, are possible if the curvature k is assumed to vary inversely as the square root of x. Thus if we take

$$\psi = (U_1\nu x)^{1/2}f(\eta),\ \eta = y(U_1/\nu x)^{1/2},\ k = (k_0/x)^{1/2}$$

we obtain $u = U_1f'$ and the momentum equation, Eq. (4), becomes

$$2f^{\text{IV}} + [f'f'' + ff'''] + C[f'^2 + 2\eta f'f''] = 0 \tag{8}$$

where

$$C = (k_0\nu/U_1)^{1/2} \tag{9}$$

and primes denote differentiation with respect to η.

The grouping C is recognized as the product of the square root of the Reynolds number based on the length x and the dimensionless local curvature kx.

In the new coordinate system the boundary conditions become

$$f(0) = 0,\quad \lim_{\eta\to-\infty} f' = 0 \tag{10}$$

and for large positive η

$$f''/f' = -C \tag{11}$$

Turbulent flow

For turbulent flow, it is first necessary to establish the eddy viscosity ϵ. In the present case, it is assumed that for moderate curvature the eddy viscosity obeys the same law as in the case where curvature is absent. Thus we take

$$\epsilon = \kappa c x U_{\max}$$

where $U_{\max}$ is the maximum velocity in the flow and the product κc is an empirical constant to be determined from the zero curvature case.[3]

It is now easily shown that similar solutions to the equations of motion, which reduce to the standard solution when $k = 0$ are possible if the curvature k is assumed to vary in-

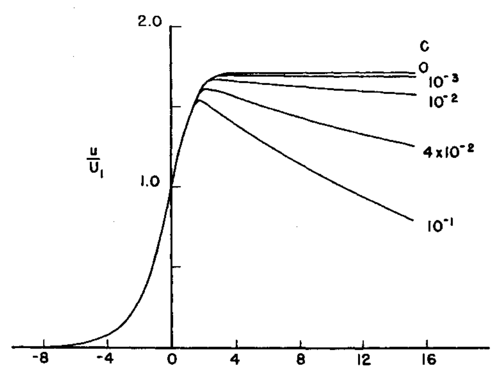

Fig. 2 Velocity profiles for curved laminar or turbulent freejet boundary.

versely as the distance x. Thus if we take

$$\psi = [1/(2)^{1/2}](x/\sigma)U_1 f,\ \eta = (2)^{1/2}(\sigma/x)y,\ k = k_0/x$$

we obtain

$$u = U_1 f'$$

and the momentum equation, Eq. (4), becomes

$$(2\kappa c U_{\max}/U_1\sigma^2)f^{\mathrm{IV}} + [f'f'' + ff'''] + C[f'^2 + 2\eta f'f''] = 0 \tag{12}$$

where now

$$C = [k_0/(2)^{1/2}\sigma] \tag{13}$$

and σ is an unspecified constant to this point. We now take $\sigma^2 = U_1/\kappa c U_{\max}$ and the equation of motion reduces to

$$2f^{\mathrm{IV}} + [f'f'' + ff'''] + C[f'^2 + 2\eta f'f''] = 0 \tag{14}$$

The constant κc is determined from the zero curvature case where $\kappa c = 1/729$. Thus in the present case

$$\sigma^2 = 729U_1/U_{\max}$$

In the turbulent case the boundary conditions are

$$f(0) = 0,\quad \lim_{\eta\to-\infty} f' = 0 \tag{15}$$

and for large positive values of η

$$f''/f' = -C \tag{16}$$

Discussion of the Problem Formulation

At this point some general discussion of the problem, as formulated previously, would seem to be in order. In the laminar case the radius of curvature is assumed to vary as $x^{1/2}$, while in the turbulent case the radius of curvature is assumed to vary as x. Each case then involves a monotonically increasing radius of curvature that is always proportional to local shear layer thickness. Physically such a flow would occur if the freejet boundary were highly curved initially, with the curvature diminishing at large distances from the origin of the jet. This is the physical situation which occurs in the case of a jet flap where the jet issues initially at some angle to the free stream but at large distances from the airfoil becomes parallel to the freestream (the radius of curvature becomes infinite). In the case of the Coanda effect the radius of curvature of the jet first decreases and then increases again as the jet approaches reattachment. Thus, if an analogy is to be made between the present solution and the Coanda effect, the present case can correspond only to the latter portion of the freejet boundary in the Coanda effect.

From a purely mathematical point of view there are possible solutions for cases other than those chosen herein. In the case of a plane jet or plane freejet boundary the conditions which determine the form of the similarity variables are[3] 1) the same functional dependence for the inertia and viscous terms; 2) conservation of momentum in the jet (since there is neither wall shear nor pressure gradient to consume momentum).

In the present case the latter condition is absent since there is an axial pressure gradient. Thus one may choose from a number of possible similarity variables. As just mentioned, the choice made in the present work was dictated by the consideration that, since the curvature is assumed small, the solution should reduce to the plane case when the curvature vanished.

The outer boundary condition, given in terms of physical variables by Eq. (6) and in terms of similarity variables by Eq. (11) [or (16)] also deserves further consideration. Murphy,[2] in considering the effects of curvature in the boundary layer, obtains this condition as the approximate representation of the irrotational external flow, consistent with the boundary-layer approximations. Since the curvature is a function only of x, Eq. (6) may be integrated to yield

$$u = g(x)e^{-ky}$$

In the case of a boundary layer $g(x)$ is generally taken to be the inviscid velocity at the surface, in the present work it may be taken as the velocity at the edge of the jet if the jet were inviscid. Employing Eq. (6) as a boundary condition then is equivalent to matching (approximately) the velocity gradient in the irrotational flow outside the viscous shear layer. Clearly in the case of small curvature and thin viscous shear layer the difference between matching the velocity and matching the velocity gradient is small.

Solution of the Equations of Motion

It is obvious that the reduced equations of motion and boundary conditions for laminar flow, Eqs. (8, 10, and 11) are the same as the reduced equations of motion and boundary conditions for the turbulent case, Eqs. (14, 15, and 16). Thus a single set of solutions to either set of equations is valid for either laminar or turbulent flow. The velocity U_1 has not been specified at this point. Without loss of generality one can take this velocity to be the velocity along the "major" streamline so that one has the additional condition $f'(0) = 1$.

One difficulty arises in the fact that the present problem is a three-point boundary condition problem with boundary conditions given at $\eta = 0$, $\eta = -\infty$, and $\eta = +\infty$.

Integration of Eq. (8) [or (14)] once yields

$$2f''' + ff'' + C\eta f'^2 = 0 \tag{17}$$

where the constant of integration has been taken to be zero to insure that the velocity and all of its derivatives vanish as η approaches $-\infty$. Further integration of Eq. (17) is carried out numerically. Starting at $\eta = 0$, with $f(0) = 0$, $f'(0) = 1$, and an assumed value of $f''(0)$, one integrates in the negative η direction to determine whether the boundary condition $f'(-\infty)$ can be satisfied for the assumed value of $f''(0)$. If not, the assumed value of $f''(0)$ is adjusted and the integration is carried out again. The procedure is repeated until the boundary condition $f'(-\infty) = 0$ is satisfied for the assumed value of $f''(0)$. Once $f''(0)$ is established the integration for $\eta > 0$ is initiated. This integration is carried out until the condition given by Eq. (16) is satisfied. From this point on the solution is obtained from the solution of Eq. (16) which can be obtained analytically.

Solutions have been obtained, following the aforementioned procedure, for values of the curvature parameter, C, from 10^{-1} to 10^{-4}.

Results

The velocity profiles obtained from the solution of Eq. (8) or (14) are presented in Fig. 2. The solution for the zero curvature case is also shown for comparison. It is particularly interesting that within a large portion of the freejet boundary the velocity profile is practically independent of the curvature, over the large range of curvature parameter considered. Similar conclusions have been reached by Sawyer[4] in his consideration of curvature effects on a jet. It is only when the velocity becomes relatively large, near the matching between the freejet boundary and the irrotational flow, that the velocity profile varies more than slightly from the zero curvature profile.

The relationship between the curvature and the pressure difference, across the freejet boundary, which causes the flow to curve is easily found. Integration of Eq. (3) for either the laminar or the turbulent case yields:

$$\frac{p(\eta_{\max}) - p(-\infty)}{\rho U_1^2} = C\int_{-\infty}^{\eta_{\max}} f'^2 d\eta \tag{18}$$

where $\eta_{\max}$ is the value of η at which the freejet boundary joins the irrotational flow. Since over most of the freejet

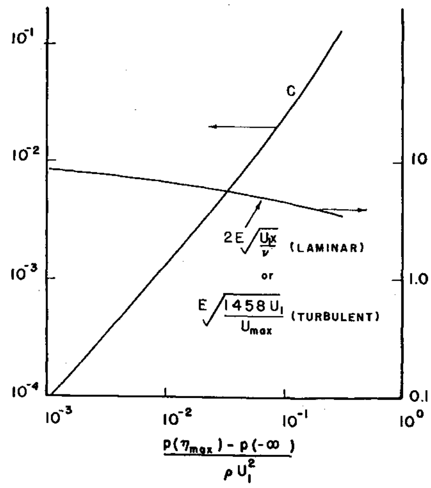

Fig. 3 Effect of lateral pressure difference on curvature and on entrainment.

boundary the velocity profile, and therefore f'^2 varies very little with curvature the integral on the right-hand side of Eq. (18) is only slightly affected by the curvature. The curvature parameter C, and therefore the curvature, varies then almost directly with the pressure difference across the freejet boundary. This is shown in Fig. 3 where the curvature parameter is shown as a function of the normalized pressure difference. As expected increasing the pressure difference increases the curvature of the flow.

Sawyer[4] in a study of reattaching jet flows, defined an entrainment parameter, E, which is a measure of the rate at which fluid is entrained by the flow. The fluid entrainment is important physically in determining the extent of the reattachment region in the Coanda effect. The entrainment parameter is defined by

$$E = \frac{1}{U_1}\frac{d}{dx}\int_{-\infty}^{y_{\max}} u dy$$

where $y_{\max}$ is the value of y corresponding to the joining of the freejet boundary and the irrotational flow. In the present analysis, one obtains for a normalized entrainment parameter in the case of laminar flow

$$2E(U_1x/\nu)^{1/2} = f(\eta_{\max}) - f(-\infty)$$

and in the case of turbulent flow

$$E(1458U_1/U_{\max})^{1/2} = f(\eta_{\max}) - f(-\infty)$$

These parameters are also presented in Fig. 3 as a function of the normalized pressure difference. It is particularly interesting to note that as the pressure difference across the jet boundary is increased the curvature of the jet boundary increases substantially while the entrainment changes only slightly and in fact decreases. The decrease is due to the alteration of the velocity profile near $\eta_{\max}$.

In the case of the curved freejet boundary then the curvature of jet is strongly influenced by the pressure difference across the jet while the entrainment is relatively insensitive to this pressure difference.

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A Matched Asymptotic Solution for Skipping Entry into Planetary Atmosphere

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I. Introduction

IN a previous paper (Ref. 1), an asymptotic solution was obtained for lifting trajectories entering a planetary atmosphere. The solution was derived by use of the method of matched asymptotic expansions, and in contrast to Refs. 2 and 3, employed only two regions: an "inner region" inside the atmosphere with dominating aerodynamic forces, and an "outer region" where gravitational forces dominate. Solutions for the "inner" and "outer" regions were matched resulting in a "composite solution" valid in both regions.

In Ref. 4, Loh classified entry trajectories as a) ballistic, b) gliding, c) skipping, and d) oscillatory paths and gave a detailed analysis of oscillatory paths. This paper is concerned with "skip paths" which are oscillatory paths of large oscillation amplitudes characterized by the fact that the vehicle leaves and re-enters the atmosphere several times before final entry. Skipping trajectories cannot be treated by the usual techniques such as those of Refs. 4 and 5, but require application of solutions which are valid both inside and outside the atmosphere. This requirement is fulfilled by the composite solution presented in Ref. 1.

The goals of this paper are 1) to show that the asymptotic approach developed by Shi and Pottsepp[1] can be extended to skipping entry trajectories, 2) to demonstrate the accuracy of results by comparing the matched asymptotic solution with the exact numerical integrations, and 3) to show that a simple closed form analytic composite solution which is uniformly valid to order unity everywhere can be used to calculate the skipping entry trajectories up to the fourth or fifth extremal points with surprisingly good accuracy. Only terms of order unity are considered in the present study.

II. Skipping Entry

Results of Ref. 1 can be extended to skipping trajectories by observing that the composite solution remains valid over

Table 1 Comparison of asymptotic expansion and numerical integrations—skipping entry

Pt #	Numerical integration I (exponential atmosphere)		Numerical integration II (isothermal atmosphere)		Asymptotic expansion (with improved constants)	
	$V'(gR)^{-1/2}$	h', ft	$V'(gR)^{-1/2}$	h', ft	$V'(gR)^{-1/2}$	h', ft
0	1.3987	360000	1.3987	360000	1.3987	360000
1	1.0378	207810	1.0374	210500	1.0325	207913
2	0.6890	1466161	0.6885	1468770	0.6839	1386558
3	0.5585	207070	0.5581	209611	0.5506	207913
4	0.3799	405842	0.3791	408254	0.3744	382755
5	0.2778	203757	0.2757	206111	0.2764	207913

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